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AN ASYMPTOTIC ESTIMATE OF THE VARIANCE OF THE SELF-INTERSECTIONS OF A PLANAR PERIODIC LORENTZ PROCESS

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ABSTRACT. We consider a $\mathbb{Z}^2$-periodic planar Lorentz process with strictly convex obstacles and finite horizon. This process describes the displacement of a particle moving in the plane with unit speed and with elastic reflection on the obstacles. We call number of self-intersections of this Lorentz process the number V_n of couples of integers (k,ℓ) smaller than n such that the particle hits a same obstacle both at the kth and at the ℓth collision times. The aim of this article is to prove that the variance of V_n is equivalent to n^2 (such a result has recently been proved for simple planar random walks in [6]).

1. INTRODUCTION

We consider a finite number of convex open sets $O_1,...,O_I\subset\mathbb{R}^2$ with boundary C^3-smooth and with non null curvature. We repeat these sets $\mathbb{Z}^2$-periodically by defining $U_{i,\ell}=O_i+\ell$ for every $(i,\ell)\in\{1,...,I\}\times\mathbb{Z}^2$. We suppose that the closures of the $U_{i,\ell}$ are pairwise disjoint. We assume that **the horizon is finite**, which means that every line meets the boundary of at least one obstacle (i.e. there is no infinite free flight). We consider a particle moving in the domain $Q:=\mathbb{R}^2\setminus\bigcup_{i=1}^I\bigcup_{\ell\in\mathbb{Z}^2}U_{i,\ell}$ with unit speed and with respect to the Descartes reflection law at its reflection times (reflected angle=incident angle). We assume that the particle starts from $[0,1[^2\cap Q$ with uniform distribution in position and in speed. The **Lorentz process** describes the evolution of the particle in Q. Because of the $\mathbb{Z}^2$-periodicity, it is strongly related to the Sinai billiard, the ergodic properties of which have been studied namely by Sinai in [9] (for its ergodicity), Bunimovich and Sinai [2, 3], Bunimovich, Chernov and Sinai [4, 5] (for central limit theorems), Young [11] (for exponential rate of decorrelation). The similarity of behaviour of the Lorentz process with a simple planar random walk has been investigated by many authors ([10, 7],...). The number of auto-intersections up to time n of a random walk $(\tilde S_n)_n$ is $\tilde V_n:=\sum_{k,\ell=1}^n\mathbf{1}_{\tilde S_k=\tilde S_\ell}$. This quantity is linked with random walks in random sceneries [1, 6]. Recently, in [6], Deligiannidis and Utev proved that $Var(\tilde V_n)\sim\tilde c n^2$ with an explicit $\tilde c$. This improved the estimation in $O(n^2\log n)$ by Bolthausen [1]. For the Lorentz process, we define $(\mathcal{I}_k,S_k)$ in $\{1,...,I\}\times\mathbb{Z}^2$ for the index of the obstacle hit at the k-th reflection time ((I_0,S_0) being the index of the obstacle at the reflection time just before time 0). Recall that $(k^{-1/2}S_k)_{k\ge1}$ admits an asymptotic positive variance matrix Σ^2. We call **number of self-intersections** of the Lorentz process up to the n-th reflection time the quantity $V_n:=\sum_{k,\ell=1}^n\mathbf{1}_{S_k=S_\ell,\mathcal{I}_k=\mathcal{I}_\ell}$. In [8], we proved that $\mathbb{E}[V_n]\sim c_0 n\log n$ with $c_0:=\frac{\sum_{i=1}^I(|\partial O_i|^2)}{(\sum_{i=1}^I|\partial O_i|)^2\pi\sqrt{\det\Sigma^2}}$, where $|\partial O_i|$ stands for the length of ∂O_i. In [8], $Var(V_n)=O(n^2\log n)$ was enough for our study of the planar Lorentz

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process in random scenery. Our proof of the following result uses decorrelation and precised local limit theorems established in [8]. It provides an alternative strategy to the one of [6].

Theorem 1. *$Var(V_n)\sim cn^2$ with*

$$c:=c_0^2\left(1+2J-\frac{\pi^2}{6}\right)\ \text{ and } J:=\int_{[0,1]^3}\frac{(1-(u+v+w))\mathbf{1}_{\{u+v+w\le 1\}}\,du\,dv\,dw}{uv+uw+vw}.$$

2. Proof of Theorem 1

Observe that the distribution of $(S_k-S_0,\mathcal{I}_k)_k$ under $\mathbb{P}$ and under $\bar\nu$ considered in [8] are the same (by $\mathbb{Z}^2$-periodicity and by construction of $\bar\nu$). We write $E_{k,\ell}:=\{S_k=S_\ell,\mathcal{I}_k=\mathcal{I}_\ell\}$. According to [8], we have

$$\mathbb{P}(E_{k,\ell})=\mathbb{P}(E_{0,|\ell-k|})=c_1|\ell-k|^{-1}+O(|\ell-k|^{-2}),\ \text{with}\ \ c_1:=\frac{\sum_i^I(\mathbb{P}(\mathcal{I}_0=i))^2}{2\pi\sqrt{\det\Sigma^2}}=\frac{c_0}{2}. \qquad (1)$$

Observe that we have $V_n=n+2\sum_{1\le k<\ell\le n}\mathbf{1}_{S_k=S_\ell,\mathcal{I}_k=\mathcal{I}_\ell}$ and so

$$Var(V_n)=4\sum_{1\le k_1<\ell_1\le n}\ \sum_{1\le k_2<\ell_2\le n}D_{k_1,\ell_1,k_2,\ell_2}=8A_1+8A_2+8A_3+4A_4,$$

with $D_{k_1,\ell_1,k_2,\ell_2}:=\mathbb{P}(E_{k_1,\ell_1}\cap E_{k_2,\ell_2})-\mathbb{P}(E_{k_1,\ell_1})\mathbb{P}(E_{k_2,\ell_2})$ and

$$A_1:=\sum_{1\le k_1<\ell_1\le k_2<\ell_2\le n}D_{k_1,\ell_1,k_2,\ell_2},\ A_2:=\sum_{1\le k_1\le k_2<\ell_1\le\ell_2\le n}D_{k_1,\ell_1,k_2,\ell_2},$$

$$A_3:=\sum_{1\le k_1<k_2<\ell_2<\ell_1\le n}D_{k_1,\ell_1,k_2,\ell_2},\ A_4:=\sum_{1\le k<\ell\le n}[\mathbb{P}(E_{k,\ell})-(\mathbb{P}(E_{k,\ell}))^2].$$

- Control of A_1.
  Due to [8], if $k_1<\ell_1\le k_2<\ell_2$, then $|D_{k_1,\ell_1,k_2,\ell_2}|\le C_1\tau_1^{k_2-\ell_1}/((\ell_1-k_1)(\ell_2-k_2))$ for some $C_1>0$ and some $\tau_1\in(0,1)$. Hence $A_1=O(n\log^2 n)=o(n^2)$.
- Control of A_4.
  Due to (1) or [10], $A_4\le C_2\sum_{1\le k<\ell\le n}(\ell-k)^{-1}=O(n\log n)=o(n^2)$.
- Control of A_2.
  According to [8], we have

$$A_2=\sum_{1\le k_1<k_2<\ell_1<\ell_2\le n}c_1^2\left[\left(\sum_x\frac{e^{-\frac{\langle(\Sigma^2)^{-1}x,x\rangle}{2}\left(\frac{1}{k_2-k_1}+\frac{1}{\ell_1-k_2}+\frac{1}{\ell_2-\ell_1}\right)}}{2\pi\sqrt{\det\Sigma^2}(k_2-k_1)(\ell_1-k_2)(\ell_2-\ell_1)}\right)-\frac{1}{(\ell_1-k_1)(\ell_2-k_2)}\right]+o(n^2), \qquad (2)$$

where $\sum_x=\sum_{x\in\mathbb{Z}^2:|x|\le||S_1||_\infty\min(k_2-k_1,\ell_1-k_2,\ell_2-\ell_1)}$ and $\langle\cdot,\cdot\rangle$ is the usual scalar product in $\mathbb{R}^2$.

  – First $A_{2,0}:=\displaystyle\sum_{1\le k_1<k_2<\ell_1<\ell_2\le n}\sum_x\frac{e^{-\frac{\langle(\Sigma^2)^{-1}x,x\rangle}{2}\left(\frac{1}{k_2-k_1}+\frac{1}{\ell_1-k_2}+\frac{1}{\ell_2-\ell_1}\right)}}{2\pi\sqrt{\det\Sigma^2}(k_2-k_1)(\ell_1-k_2)(\ell_2-\ell_1)}$

$$=\sum_{(k_1,m_0,m_1,m_2)\in E_n}\ \sum_{|x|\le||S_1||_\infty\min(m_0,m_1,m_2)}\frac{e^{-\frac{\langle(\Sigma^2)^{-1}x,x\rangle}{2}\left(\frac{1}{m_0}+\frac{1}{m_1}+\frac{1}{m_2}\right)}}{2\pi\sqrt{\det\Sigma^2}m_0m_1m_2},$$

with $E_n := \{(k_1, m_0, m_1, m_2) \in \mathbb{Z}_+ \,:\, k_1 + m_0 + m_1 + m_2 \le n\}$. Observe that, using a comparison series-integral, we obtain

$$\sup_{||S_1||_\infty \le a \le 3||S_1||_\infty} \left| \sum_{x\in\mathbb{Z}^2 \,:\, |x|\le am} e^{-\frac{\langle(\Sigma^2)^{-1}x,x\rangle}{2m}} - 2\pi m\sqrt{\det\Sigma^2} \right| = O(\sqrt{m}). \tag{3}$$

So $$A_{2,0} = \sum_{(k_1,m_0,m_1,m_2)\in E_n} \frac{1+O(\min(m_0,m_1,m_2)^{-1/2})}{m_0m_1+m_0m_2+m_1m_2} \sim n^2 J.$$

– Second $\sum_{1\le k_1<k_2<\ell_1<\ell_2\le n} \frac{1}{(\ell_1-k_1)(\ell_2-k_2)} = A_{2,1} + 2A_{2,2}$,

with $$A_{2,1} := \sum_{k=1}^{n} \sum_{\max(1,2k-n)\le m\le k} \frac{n-(2k-m)+1}{k^2} \le \sum_{k=1}^{n}\sum_{m=0}^{k} \frac{n}{k^2} = O(n\log n) = o(n^2),$$

and $$A_{2,2} := \sum_{1\le k<\ell\le n} \sum_{\max(0,k+\ell-n)\le m\le k} \frac{n-(k+\ell-m)+1}{k\ell}$$

$$\begin{aligned}
&= \sum_{\ell=1}^{\lfloor n/2\rfloor}\sum_{k=1}^{\ell-1}\sum_{m=0}^{k}\cdots + \sum_{\ell=\lfloor n/2\rfloor+1}^{n}\sum_{k=1}^{n-\ell}\sum_{m=0}^{k}\cdots + \sum_{\ell=\lfloor n/2\rfloor+1}^{n}\sum_{k=n-\ell+1}^{\ell-1}\sum_{m=k+\ell-n}^{k}\cdots \\
&= o(n^2) + \sum_{\ell=1}^{\lfloor n/2\rfloor}\sum_{k=1}^{\ell-1}\frac{2(n-\ell)-k}{2\ell} + \sum_{\ell=\lfloor n/2\rfloor+1}^{n}\sum_{k=1}^{n-\ell}\frac{2(n-\ell)-k}{2\ell} + \sum_{\ell=\lfloor n/2\rfloor+1}^{n}\sum_{k=n-\ell+1}^{\ell-1}\frac{(n-\ell)^2}{2k\ell} \\
&= o(n^2) + \sum_{\ell=1}^{\lfloor n/2\rfloor}\frac{4n-5\ell}{4} + \sum_{\ell=\lfloor n/2\rfloor+1}^{n}\frac{3(n-\ell)^2}{4\ell} + \sum_{\ell=\lfloor n/2\rfloor+1}^{n}\frac{(n-\ell)^2}{2\ell}\log\left(\frac{\ell}{n-\ell}\right) \\
&\sim n^2\left(-\frac{1}{8}+\frac{3}{4}\log 2+\frac{I}{2}\right),
\end{aligned}$$

with

$$\begin{aligned}
I &:= \int_{1/2}^{1} \frac{(1-u)^2}{u}\log\left(\frac{u}{1-u}\right)du \\
&= \left[Li_2(u) + \frac{1}{2}\left(u+\log u(u^2+\log u-4u)+\log(1-u)(-u^2+4u-3)\right)\right]_{1/2}^{1},
\end{aligned}$$

with $Li_2(z) := \sum_{k\ge 1}\frac{z^k}{k^2}$. So $I = Li_2(1) - Li_2(1/2) + \frac{1}{4} - \frac{\log^2 2}{2} - \frac{3}{2}\log 2 = \frac{\pi^2}{6} - (\frac{\pi^2}{12} - \frac{\log^2 2}{2}) + \frac{1}{4} - \frac{\log^2 2}{2} - \frac{3}{2}\log 2 = \frac{\pi^2}{12} + \frac{1}{4} - \frac{3}{2}\log 2$. Hence we have $A_{2,1} + 2A_{2,2} \sim \frac{\pi^2}{12}n^2$.

- Control of A_3.

Notice that $\sum_{1\le k_1<k_2<\ell_2<\ell_1\le n} \mathbb{P}(E_{k_1,\ell_1}\cap E_{k_2,\ell_2})$ and $\sum_{1\le k_1<k_2<\ell_2<\ell_1\le n} \mathbb{P}(E_{k_1,\ell_1})\mathbb{P}(E_{k_2,\ell_2})$ are in $n^2\log n$. But we will see that their difference is in n^2. According to [8] and to (1), we have:

$$A_3 = c_1 \sum_{1\le k_1\le k_2<\ell_2<\ell_1\le n} \left[\left(\sum_x \frac{e^{-\frac{\langle(\Sigma^2)^{-1}x,x\rangle}{2}\left(\frac{1}{k_2-k_1}+\frac{1}{\ell_1-\ell_2}\right)}}{2\pi\sqrt{\det\Sigma^2}(k_2-k_1)(\ell_1-\ell_2)}\right) - \frac{1}{(\ell_1-k_1)}\right]\mathbb{P}(E_{k_2,\ell_2}) + o(n^2) \tag{4}$$

with the same notations as for (2). Using again (3) and (1), we obtain

$$\begin{aligned} A_3 &= o(n^2) + c_1^2 \sum_{1\le k_1<k_2<\ell_2\le \ell_1\le n} \frac{1}{\ell_2-k_2}\left[\frac{1}{(\ell_1-k_1)-(\ell_2-k_2)} - \frac{1}{(\ell_1-k_1)}\right] \\ &= o(n^2) + c_1^2 \sum_{1\le k_1<k_2<\ell_2\le \ell_1\le n} \frac{1}{(\ell_1-k_1)[(\ell_1-k_1)-(\ell_2-k_2)]} \\ &\sim c_1^2 n^2 \int_{[0,1]^4} \frac{\mathbf{1}_{\{t+u+v+w<1\}}\, dt\, du\, dv\, dw}{(u+w)(u+v+w)} = c_1^2 n^2 \int_{0\le u\le r\le s\le 1} \frac{(1-s)\, du\, dr\, ds}{rs} = \frac{c_1^2}{2} n^2. \end{aligned}$$

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